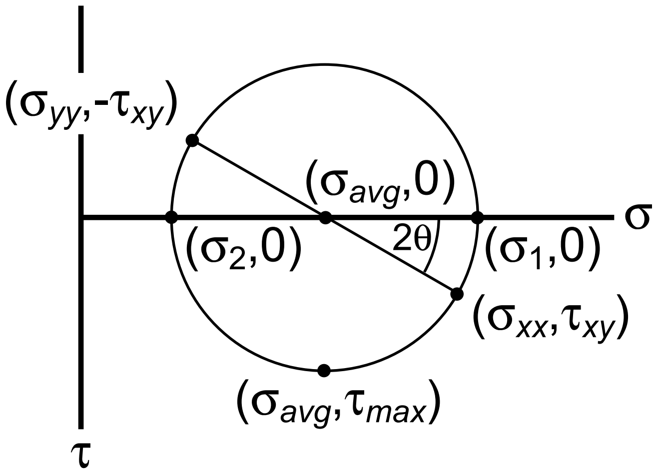
Principal Stresses: Mohr's Circle

2D:



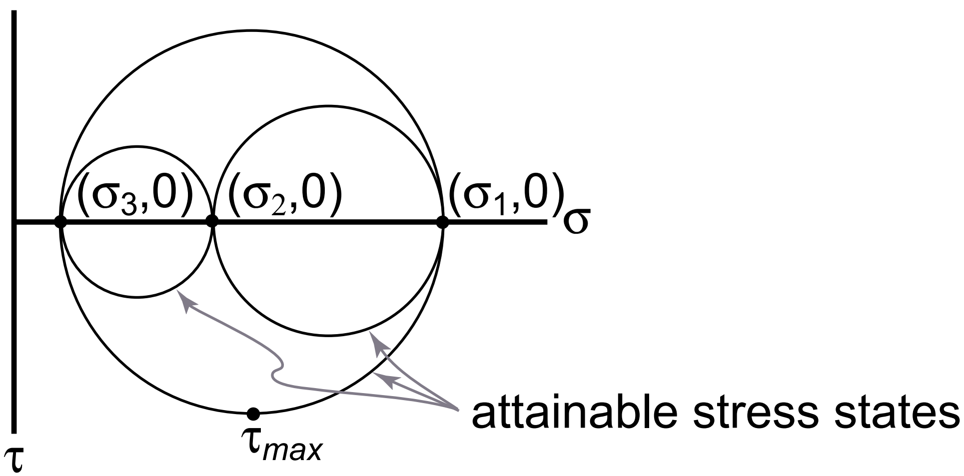
$$\sigma_{avg} = \frac{\sigma_{xx}+\sigma_{yy}}{2}$$

$$\tan(2\theta) = \frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}}$$

$$\tau_{max} = \frac{\sigma_1-\sigma_2}{2} = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^{\,2}}$$

$$\sigma_1,\sigma_2 = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^{\,2}}$$

3D:



Elasticity

Tensor notation:	11	22	33	23=32	13=31	12=21
Matrix notation:	1	2	3	4	5	6

$$\begin{bmatrix} \varepsilon_{xx} \equiv \varepsilon_1 \\ \varepsilon_{yy} \equiv \varepsilon_2 \\ \varepsilon_{zz} \equiv \varepsilon_3 \\ \gamma_{yz} \equiv \varepsilon_4 \\ \gamma_{xz} \equiv \varepsilon_5 \\ \gamma_{xy} \equiv \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & \frac{-\nu_{yx}}{E_y} & \frac{-\nu_{zx}}{E_z} & \frac{\eta_{yz,x}}{G_{yz}} & \frac{\eta_{xz,x}}{G_{xz}} & \frac{\eta_{xy,x}}{G_{xy}} \\ \frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{-\nu_{zy}}{E_z} & \frac{\eta_{yz,y}}{G_{yz}} & \frac{\eta_{xz,y}}{G_{xz}} & \frac{\eta_{xy,y}}{G_{xy}} \\ \frac{-\nu_{xz}}{E_x} & \frac{-\nu_{yz}}{E_y} & \frac{1}{E_z} & \frac{\eta_{yz,z}}{G_{yz}} & \frac{\eta_{xz,z}}{G_{xz}} & \frac{\eta_{xy,z}}{G_{xy}} \\ \frac{\eta_{x,yz}}{E_x} & \frac{\eta_{y,yz}}{E_y} & \frac{\eta_{z,yz}}{E_z} & \frac{1}{G_{yz}} & \frac{\mu_{xz,yz}}{G_{xz}} & \frac{\mu_{xy,yz}}{G_{xy}} \\ \frac{\eta_{x,xz}}{E_x} & \frac{\eta_{y,xz}}{E_y} & \frac{\eta_{x,xz}}{E_z} & \frac{\mu_{yz,xz}}{G_{yz}} & \frac{1}{G_{xz}} & \frac{\mu_{xy,xz}}{G_{xy}} \\ \frac{\eta_{x,xy}}{E_x} & \frac{\eta_{y,xy}}{E_y} & \frac{\eta_{z,xy}}{E_z} & \frac{\mu_{yz,xy}}{G_{yz}} & \frac{\mu_{xz,xy}}{G_{xz}} & \frac{1}{G_{xy}} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{xx} \equiv \sigma_1 \\ \sigma_{yy} \equiv \sigma_2 \\ \sigma_{zz} \equiv \sigma_3 \\ \sigma_{yz} \equiv \sigma_4 \\ \sigma_{xz} \equiv \sigma_5 \\ \sigma_{xy} \equiv \sigma_6 \end{bmatrix}$$

Note: $\nu_{12} = -\frac{\varepsilon_{22}}{\varepsilon_{11}}$ $\eta_{12,1} = \frac{\varepsilon_{11}}{\gamma_{12}}$ $\eta_{1,12} = \frac{\gamma_{12}}{\varepsilon_{11}}$ $\mu_{12,13} = \frac{\gamma_{13}}{\gamma_{12}}$ $S_{21} = \frac{-\nu_{12}}{E_1}$ (b/c $\varepsilon_j = S_{ji}\sigma_i$)

Elasticity

Crystal symmetry reduces the compliance tensor further (see Nye, 1985)…

e.g., cubic symmetry (or isotropic)

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{44} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{44} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

Elasticity

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1-\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{bmatrix}$$

3 main symmetries:

Crystal	Material (Polycrystal)
Cubic	isotropic
Tetragonal	transversely isotropic
Orthorhombic	orthotropic

Orthotropy

- 3 orthogonal planes of symmetry.
- If the three normal axes are reversed, there is no change in $C \Longrightarrow$ invariant.
- $C_{14}=C_{15}=C_{16}=C_{24}=C_{25}=C_{26}=C_{14}=C_{34}=C_{35}=C_{36}=C_{45}=C_{46}=C_{56}=0$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{32} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \Longrightarrow \text{9 independent coefficients}$$

Orthotropy

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} = \frac{1}{E_1} & S_{21} = \frac{-\nu_{21}}{E_2} & S_{31} = \frac{-\nu_{31}}{E_3} & 0 & 0 & 0 \\ S_{12} = \frac{-\nu_{12}}{E_1} & S_{22} = \frac{1}{E_2} & S_{32} = \frac{-\nu_{32}}{E_3} & 0 & 0 & 0 \\ S_{13} = \frac{-\nu_{13}}{E_1} & S_{23} = \frac{-\nu_{23}}{E_2} & S_{33} = \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} = \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} = \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} = \frac{1}{G_{12}} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

$\nu_{ij} \neq \nu_{ij}$; but due to symmetry, $E_1\nu_{21} = E_2\nu_{12}$, $E_2\nu_{32} = E_3\nu_{23}$, and $E_3\nu_{13} = E_1\nu_{31}$.

Orthotropy

Therefore,

$$C_{11} = \frac{S_{22}S_{33} - S_{23}^2}{\Delta} = \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta'} \quad C_{12} = \frac{S_{13}S_{23} - S_{12}S_{33}}{\Delta} = \frac{\nu_{21}+\nu_{31}\nu_{23}}{E_2E_3\Delta'} \quad C_{44} = \frac{1}{S_{44}} = G_{23}$$

$$C_{22} = \frac{S_{11}S_{33} - S_{13}^2}{\Delta} = \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta'} \quad C_{13} = \frac{S_{12}S_{23} - S_{13}S_{22}}{\Delta} = \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta'} \quad C_{55} = \frac{1}{S_{55}} = G_{13}$$

$$C_{33} = \frac{S_{11}S_{22} - S_{12}^2}{\Delta} = \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta'} \quad C_{23} = \frac{S_{12}S_{13} - S_{23}S_{11}}{\Delta} = \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta'} \quad C_{66} = \frac{1}{S_{66}} = G_{12}$$

where, $\Delta = Det\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$ $\quad \Delta' = \dfrac{1-\nu_{21}\nu_{12}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$

Finally,

$$E_{11} = C_{11} + \frac{2C_{12}C_{13}C_{23} - C_{12}^2C_{33} - C_{13}^2C_{22}}{C_{22}C_{33} - C_{23}^2} \qquad \nu_{12} = \frac{C_{12}C_{33} - C_{13}C_{23}}{C_{22}C_{33} - C_{23}^2}, \text{ etc.}$$

Note: If (C_{11}=C_{22}=C_{33}, C_{12}=C_{23}=C_{13}), these can be reduced to an isotropic material.

Orthotropy

Recall transformations using indicial notation: $\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl}$ $\varepsilon'_{ij} = a_{ik} a_{jl} \varepsilon_{kl}$

$C'_{ijkl} = a_{im} a_{jn} a_{ko} a_{lp} C_{mnop}$

Using matrix notation: 3x3 σ: $\sigma' = a \cdot \sigma \cdot a^T$ 3x3 a

1x6 σ: $\sigma' = T_\sigma \cdot \sigma$ 6x6 T_σ

where,

$$T_\sigma = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{12}a_{13} & 2a_{11}a_{13} & 2a_{11}a_{12} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{22}a_{23} & 2a_{21}a_{23} & 2a_{21}a_{22} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{32}a_{33} & 2a_{31}a_{33} & 2a_{31}a_{32} \\ a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & a_{22}a_{33} + a_{23}a_{32} & a_{23}a_{31} + a_{21}a_{33} & a_{31}a_{22} + a_{21}a_{32} \\ a_{11}a_{31} & a_{12}a_{32} & a_{13}a_{33} & a_{13}a_{32} + a_{12}a_{33} & a_{11}a_{33} + a_{13}a_{31} & a_{11}a_{32} + a_{12}a_{31} \\ a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & a_{12}a_{23} + a_{13}a_{22} & a_{11}a_{23} + a_{13}a_{21} & a_{11}a_{22} + a_{12}a_{21} \end{bmatrix}$$

1x6 ε: $\varepsilon' = T_\varepsilon \cdot \varepsilon$ 6x6 T_ε $\Rightarrow$ move '2's in upper right to lower left

Note: $T_\varepsilon \equiv b$ (Bowman)

Therefore, $C' = T_\sigma \cdot C \cdot T_\sigma^T$ and $S' = T_\varepsilon \cdot S \cdot T_\varepsilon^T$